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Review Article

The Distribution Properties of Two-Parameter Exponential Distribution Order Statistics

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Abstract

This paper proposes the distribution function and density function of double parameter exponential distribution and discusses some important distribution properties of order statistics. We prove that random variables following the double parameter exponential type distribution X_1, X_2,..., X_n are not mutually independent and do not follow the same distribution, but that the X_i, X_j meet the dependency of TP_2 to establish RTI ($X_i | X_j$), LTD ($X_i | X_j$) and *RSCI*.

Keywords: Order statistics; Double parameter exponential distribution; TP_2; *RTI*; *LTD*; *RSCI*

1. Introduction

In this paper, some important properties of order statistics of two-parameter exponential distribution are discussed when the distribution and density functions of a two-parameter distribution is given. We also proved that the random variables X_1, X_2,..., X_n, obeying the two-parameter exponential distribution are not independent of each other, and do not obey the same distribution. Order statistics is a kind of statistics distribution commonly used in statistical theory and application of which there are many research [1-6]. The two parameter exponential distribution is also a very useful component in reliability engineering. This study considers the nature of order statistics. Its density function and distribution functions are respectively [7];

$$f(x) = \begin{cases} \frac{1}{\beta}\exp(-\frac{x-\alpha}{\beta}), x \geq \alpha \\ 0, x < \alpha \end{cases}$$

$$F(x) = \begin{cases} 1-\exp(-\frac{x-\alpha}{\beta}), x \geq \alpha \\ 0, x < \alpha \end{cases}$$

For all $\alpha \in R, \beta > 0$

2. Prerequisite Knowledge

2.1 Lemma 1

Let all X follow a continuous distribution function $F(x)$ and its density function of $F(x)$, $\{a < x < b\}$, $X_1, X_2,...X_n$ is a simple random sample with acapacity of N from X_2

1. The joint probability density function of $(X_1, X_2,..., X_n)$ *if* $a \leq X_1 < X_2 < ... < X_n \leq b$ is

$$g_{1,2,...,n}(X_1, X_2,..., X_n) = n!\prod_{i=1}^{n} f(x_i) \quad (1)$$

Otherwise,

$g_{1,2,...,n}(X_1, X_2, ..., X_n)$ 0

2. The joint probability density function of order statistic $(Xi, Xj)(1 \leq I \leq j \leq n)$ is a $\leq x \leq y \leq b$ and

$$g_{i,j}(x, y) = \frac{n! f(x) f(y)}{(i-1)!(j-i-1)!(n-j)!}[F(y)-F(x)]^{i-1}[1-F(y)]^{n-j} \quad (2)$$

Otherwise,

$g_{i,j}(x, y)$ 0

3. The probability density function of order statistics *X (k)* is

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x), a < x < b \qquad (3)$$

In particular, when $k = 1$, there is

$$f_1(x) = n(1-F(x))^{n-1} f(x), a < x < b \qquad (4)$$

When $k = n$, there is

$$f_n(x) = nF^{n-1}(x) f(x), a < x < b. \qquad (5)$$

2.2 Lemma 2

Assume that X and Y are two random variables, the joint probability is $f(x)$ if the inequality $x_1 \le x_2$, $y_1 \le y_2$ is satisfied [10].

There is; $\begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \ge 0$, and we say X, Y is TP2 dependent

2.3 Lemma 3

For a fixed x, y, if $P(X > x, Y > y \mid X > x^1, Y > y^1)$ is a monotonic increasing function for variables x^1 and y^1, then variables X, Y satisfies RSCI [10] .

2.4 Lemma 4

For any y_1, if $P(Y \le y_1 \mid X \le x_1)$ is monotone decreasing function of x_1, then Y is the left tail decreasing function of X, denoted by *Ltd* (*XY*) [11].

2.5 Lemma 5

For any y_1, if $P(Y > y_1 \mid X > x_1)$ is an incremental function of x_1, then Y is the right tail growth of X, denoted by *RTI* (*XY*) [11].

3. Main Conclusion

3.1 Theorem 1

Let the total X follow a continuous distribution function of $F(x)$ and its density function $F(x)$ $(a < x < b)$, $X_1, X_2, \ldots, X_n$ be a simple random sample with a capacity of N from the population X, and $X_1, X_2, \ldots, X_n$

1. The order statistics of the joint probability density function of (X_1, X_2,..., X_n) is as follows; $a \leq X_1 < X_2 < \ldots < X_n \leq b$

If then

$$g_{1,2,\ldots,n}(X_1, X_2, \ldots, X_n) = \begin{cases} n!(\frac{1}{\beta})^n \exp(\sum_{i=1}^{n}(-\frac{x_1-\alpha}{\beta})), x \geq \alpha \\ 0, x < \alpha \end{cases} \quad (6)$$

2. The probability density function of X_k is;

$$f_k(x) = \frac{n!}{\beta(k-1)!(n-k)!}[\exp(\frac{2\alpha}{\beta} - \frac{2x}{\beta})]^{n-k+1}[1-\exp(-\frac{x-\alpha}{\beta})]^{k-1}, x \geq \alpha \quad (7)$$

In particular, when k 1, there is

$$f_1(x) = \frac{n}{\beta}[\exp(-\frac{x-\alpha}{\beta})]^n, x \geq \alpha \quad (8)$$

When $k= n$,

$$f_n(x) = \frac{n}{\beta}[1-\exp(-\frac{x-\alpha}{\beta})]^{n-1} \exp(-\frac{x-\alpha}{\beta}), x \geq \alpha. \quad (9)$$

Which is proved from lemma 1.

3.2 Theorem 2

If X_1, X_2, ..., X_n is independent of each other and obeys a two-parameter exponential distribution, then X_1, X_2, ..., X_n is not independent of each other and does not obey the same distribution.

Proof : Let $n = 2$ be used to represent the observations of x_1 and x_2 with X_1 and X_2 respectively. It can be seen from lemma 1 that the density function of (X_1, X_2) is

$$f_{X_1,X_2}(x_1, x_2) = 2!\prod_{i=1}^{2} f(x_i) = 2f(x_1)(x_2),$$

The density functions of X_1 and X_2 are respectively

$$f(x_1) = 2[1-F(x_1)]f(x_1), 0 < x_1 < +\infty,$$

$$f(x_2) = 2F(x_2)f(x_2), 0 < x_2 < +\infty.$$

Assuming that X_1 and X_2 are independent, then $f_{X1,X2}(x_1, x_2) = f(x_1).f(x_2)$, and $2f(x_1)f(x_2) = 2[1-F(x_1)]f(x_1).2F(x_2)f(x_2) = 4[1-F(x_1)]F(x_2)f(x_1)f(x_2)$.

Therefore,

$$[1-F(x_1)]F(x_2)=\frac{1}{2},$$

Then,

$$\exp(-\frac{x_1-\alpha}{\beta}).[1-\exp(-\frac{x_2-\alpha}{\beta})]=\frac{1}{2}$$

Which obviously does not hold. So

$$f_{X1,X2}(x_1, x_2) \neq f(x_1).f(x_2)$$

Therefore, X_1 and X_2 are not independent of each other and do not obey the same distribution. Hence proved.

3.3 Theorem 3

If $X_1, X_2,..., X_n$ are independent of each other and obey the two-parameter exponential distribution, then X_i, Y_j $(I < j)$ is *TP*2 dependent.

Proof: The observations of *xi* and *xj* are represented by *Xi*, *Xj*. For any $x_i < x_j$, the joint density function of Lemma 1, is

$$f_{i,j}(x_i,x_j)=\begin{cases} C[F(x_i)]^{i-1}[1-F(x_j)]^{n-j}f(x_i)f(x_j), 0<x_i\le x_j \\ 0, otherwise\end{cases}$$

including :

$$C=\frac{n!}{(i-1)!(j-1-1)!(n-j)!}$$

Which is a constant independent of x_i, x_j. When $0<x_1\le x_2, 0<y\le y_2$, there exist

$$f_{i,j}(x_1,y_1)f_{i,j}(x_2,y_2)=C^2[F(x_1)]^{i-1}[1-F(y_1)]^{n-j} \text{ X}$$
$$f(x_1)f(y_1)[F(x_2)]^{i-1}[F(y_2)-F(x_2)]^{j-i-1}[1-F(y_2)]^{n-j}f(x_2)f(y_2),$$

$$f_{i,j}(x_1, y_2) f_{i,j}(x_2, y_1) = C^2 [F(x_1)]^{i-1} [F(y_2) - F(x_1)]^{j-i-1} [1 - F(y_2)]^{n-j}$$
$$(f(x_1) f(y_2) [F(x_1)]^{i-1} [F(y_2) - F(x_1)]^{j-i-1} [1 - F(y_2)]^{n-j} f(x_1) f(y_2)]).$$

Thus we need to proof that

$$f_{i,j}(x_1, y_1) f_{i,j}(x_2, y_2) \geq f_{i,j}(x_1, y_2) f_{i,j}(x_2, y_1) .$$

We only need

$$[F(y_1) - F(x_1)][F(y_2) - F(x_2)] \geq [F(y_1) - F(x_2)][F(y_2) - F(x_1)],$$

Simply;

$$F(y_1)[F(x_2) - F(x_1)] \leq F(y_2)[F(x_2) - F(x_1)].$$

Again because $0 < x_1 \leq x_2, 0 < y_1 \leq y_2$, therefore $F(x_2) \geq F(x_1), F(y_2) \geq F(y_1)$.

So $[F(y_1) - F(x_1)][F(y_2) - F(x_2)] \geq [F(y_1) - F(x_2)][F(y_2) - F(x_1)]$.

Therefore $f_{i,j}(x_1, y_1) f_{i,j}(x_2, y_2) \geq f_{i,j}(x_1, y_2) f_{i,j}(x_2, y_1)$

So $X_i, Y_j (i < j)$ is dependent of $TP2$.

3.4 Theorem 4

Suppose $X_1, X_2, ..., X_n$ is *n* independent and identically distributed random variables from the exponential distribution of two parameters, then *RTI* $(X_j \mid X_i)$, $i < j$. It is proved that for any *s, t* when $s > t$, there is

$$P(X_j > t \mid X_i > s) = \frac{P(X_j > t, X_i > s}{P(X_i > s)} = \frac{P(X_i > s)}{P(X_i > s)} = 1$$

When $s < t$, there is

$$P(X_j > t \mid X_i > s) = \frac{P(X_j > t, X_i > s)}{P(X_i > s)} = \frac{P(X_i > s) - P(s \leq X_j \leq t, X_i > s}{P(X_i > s)} = 1 - \frac{P(s \leq X_j \leq t, X_i > s)}{P(X_i > s)} \leq 1$$

When *it* is fixed and *so* increases. The integral of $P(\leq X_j \leq t, Xs)$ decreases $P(s \leq X_j \leq t, X_i > s)$

$P(X_j > t \mid Xi > s)$ $LTD(Xi \mid X_j)$, $i < j$ according to the density functions becomes smaller and becomes larger.

3.5 Theorem 5

Suppose X_i and X_j are two random variables independently obeying the same two-parameter exponent. Then X_1 and X_2 satisfy *RCSI* (i.e for fixed *X* and *Y* and *Y*, if $P(X x, Y y \mid X x', Y y')$ is a monotonic increasing function of variables x' and *y'*, it is proved that the joint density functions of X_1 and X_2 is $f_{1,2}(x_1, x_2)$.

i. When $x_1 < x'_1, x_2 < x'_2$, there is

$$P(X_1 > x_1, X_2 > x_2 | X_1 > x', X_2 > x'_2) =$$

$$\frac{P(X_1 > x_1, X_2 > x_2, X_1 > x'_1, X_2 > x'_2)}{P(X_1 > x'_1, X_2 > x'_2)} = \frac{P(X_1 > x'_1, X_2 > x'_2)}{P(X_1 > x'_1, X_2 > x'_2)} = 1$$

When x_1, x_2 remain unchanged, $P(X_1 > x_1, X_2 > x_2 | X_1 > x_1, X_2 > x_2)$ is constant.

ii. When $x_1 > x_1, x_2 > x_2$, there are

$$P(X_1 > x_1, X_2 > x_2 | X_1 > x_1, X_2 > x_2) =$$

$$\frac{P(X_1 > x_1, X_2 > x_2, X_1 > x'_1, X_2 > x'_2)}{P(X_1 > x'_1, X_2 > x'_2)} = \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x'_1, X_2 > x'_2)} < 1$$

For the order of $\overline{F_{1,2}}(s,t) = P(X_1 > s, X_2 > t)$, when $s < t$, their joint density functions are;

$$\overline{F}_{1,2}(s,t) = \iint_{x>s,y>t} \frac{2}{\beta^2} \exp(-\frac{x-|\alpha}{\beta} - \frac{y-\alpha}{\beta}) dxdy$$

$$= 2\int_s^{+\infty} \frac{1}{\beta} \exp(-\frac{x-\alpha}{\beta}) dx \int_t^{+\infty} \frac{1}{\beta} \exp(-\frac{y-\alpha}{\beta}) dy$$

$$= \exp(-\frac{s-\alpha}{\beta}) \exp(-\frac{t-\alpha}{\beta})$$

$$= \exp(-\frac{s-\alpha}{\beta} - \frac{t-\alpha}{\beta})$$

$$= \exp(\frac{2\alpha}{\beta} - \frac{s+t}{\beta})$$

Therefore;

$$\frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1, X_2 > x_2)} = \frac{\exp(\frac{2\alpha}{\beta} - \frac{x_1 + x_2}{\beta})}{\exp(\frac{2\alpha}{\beta} - \frac{x_1 + x_2}{\beta})} = \exp(\frac{x'_1 + x'_2}{\beta} - \frac{x_1 + x_2}{\beta})$$

When x_1, x_2 is invariant, $P(X_1 > x_1, X_2 > x_2 | X_1 > x'_1, X_2 > x'_2)$, x'_1, x'_2 is increasing function

iii When $x_1 < x'_1, x_2 > x'_2$ or $x_1 > x'x, x_2 < x'_2$ are equally available,

$$P(X_1 > x_1, X_2 > x_2 | X_1 > x'_1, X_2 > x'_2) = \exp(\frac{x'_2 - x_2}{\beta})$$

Or

$$P(X_1 > x_1, X_2 > x_2 | X_1 > x'_1, X_2 > x'_2) = \exp(\frac{x_1' - x_1}{\beta})$$

When x_1, x_2 are fixed, they are incremental functions of x'_1, x'_2. In conclusion, $P(X_1 > x_1, X_2 > x_2 \mid X_1 > x'_1, X_2 > x'_2)$ are incremental functions of x_1', x'_2. So X_1, X_2 satisfies *RSCI*.

4. Conclusion

Some distribution properties of order statistics obeying two-parameter exponential distribution are discussed. It is proved that when X_1, X_2,..., X_n are independent of each other and obey the exponential distribution of two-parameters, the order statistics X_1, X_2, ..., X_n is not independent of each other and does not obey the same distribution, but X_i, X_j satisfies *TP2* dependence. For any $i < j$, there are *RTI* ($X_j \mid X_i$), *LTD*($X_i \mid X_j$), and X_1, X_2 that satisfy *RSCI*.

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